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UNCERTAINTY AND DISCRETE MAXIMIN

© V. I. Zhukovskiy

Moscow State University named after Lomonosov
Faculty of Computational Mathematics and Cybernetics
Department of Optimal Control
Leninskiye Gory, GSP-1, Moscow, 119991, Russia
e-mail: *zhkvlad@yandex.ru*

© L. V. Smirnova

State University of Humanities and Technology
Zelenaya, 22, Orekhovo-Zuevo, 142611, Russia
e-mail: *smirnovalidiya@rambler.ru*

Uncertainty and discrete maximin.

Zhukovskiy V. I., Smirnova L. V.

Abstract. The article consists of two parts. The first part is devoted to general questions that are related to uncertainty: causes and sources of uncertainties appearance, classification of uncertainties in economic systems and approach to their assessment. In the second part the concept of maximin, based on the principle of guaranteed result (Wald's principle) is considered. In this case, maximin is interpreted from viewpoint of two-level hierarchical game. On the basis of the maximin concept, a guaranteed solution in outcomes for K-stage positional single-criterion linear quadratic problem under uncertainty is formalized. An explicit form of the guaranteed solution for this problem is found.

Keywords: *Nash equilibrium, Berge equilibrium, uncertainty, maximin, difference (multi-stage) system.*

Introduction

The genealogical tree of game theory has roots going deep into centuries[1], powerful trunks and a thick crown in which numerous modern works on game theory are intertwined. A flowering and fruitful trunk — noncooperative games — was cultivated in 1949 by twenty-one-year-old American mathematician John Nash. In his 27 pages-long doctoral dissertation defended at Princeton University, Nash managed to separate out a «new face» of competition and defined a strategy profile, which was later called Nash equilibrium. After 45 years, J. Nash together with R. Selten and J. Harsanyi were awarded the Nobel Prize in Economic Sciences «for their pioneering analysis of equilibria

[1] "The need for making decisions under conflict is as old as humanity itself." A quote from [1, p. 10].

in the theory of non-cooperative games.» However, the concept of Nash equilibrium has a number of negative properties (internal instability, non-uniqueness, no equivalence, no interchangeability, improvability) [2, Section 2.2.3] and, especially striking, selfishness that permeates it. Really, following the concept of Nash equilibrium, each conflicting party seeks to improve only his result, paying no attention to the interests of others. In [3], we make an attempt to plant a new sprout, dictated by the altruism of the Golden Rule of ethics-the aspiration to help others, sometimes forgetting about oneself. The life-giving rain for this blossom to flourish is triggered by the following factors.

First, an integration of dynamic programming with the Lyapunov function method was proposed by Academician N.N. Krasovskii. As a result, Lyapunov's brilliant idea to perform the stability analysis of the trajectories of a differential equation using only the definiteness of Lyapunov functions was transformed into the ability to find equilibrium strategies (in particular, Berge equilibrium) by the extreme properties of Bellman–Krasovskii functions.

Second, optimal solutions of guaranteeing control problems are unstable with respect to small disturbances and informational errors. In view of this fact, for regularization of optimal solutions, Academician Krasovskii and his followers introduced and developed the ideology of control procedures in which a real object is considered jointly with a similar reference system–guide. The motion of a guide, conceivable or modeled on a computer, acts as an ideal undisturbed process. Actually, this leads to a stabilization problem in a new game-theoretic statement. In the late 1970s, the control concept of differential and evolutionary systems based on a joint consideration of a real controlled object and an auxiliary model system (guide) was further refined. A convenient tool on that way was a uniform description of the dynamics of a model system suggested by Krasovskii. Guiding control will be adopted to identify a class of differential positional games for which there exists a Berge equilibrium in a corresponding differential positional game with «separated» dynamics.

Third, due to the conceptual specifics of Berge equilibrium, the Germeier convolution of the players' payoff functions can be successfully applied not only in the static, but also in the dynamic case of the Golden Rule of ethics.

And *fourth*, in mathematical models the presence of uncertain factors (uncertainties) without any probabilistic characteristics, just known ranges (e.g., price jumps in a sales market, disruption and (or) variations in the nomenclature of supplies, man-made changes, etc.), and also multistage control (control at discrete time instants) were successfully taken into account. (As a matter of fact, many problems of economic planning, engineering and production control, military science, ecology, medicine are described by difference equations: in practice, information on the state of a process is acquired and the process itself is controlled at discrete time instants).

1. Uncertainty and its types

1.1. **Causes of uncertainty.** In the study of any system, including economic ones, the uncertainties affecting it have to be taken into account.

First, this is due to the peculiarities of the evolution of weakly structured systems-the systems described by both qualitative and quantitative characteristics with dominating qualitative, little-known or uncertain parameters.

Second, economic systems are controlled under insufficient knowledge of the state of an external environment, often with large investments of resources. Moreover, a special class of problems is to study economic systems that will operate at their limiting capability, in order to obtain maximum economic or any other benefits.

Third, the need to consider uncertainty becomes vital if separate, often conflicting subsystems are included into a system under study. In this case, an ambiguous solution cannot be found, and some kind of compromise has to be reached accordingly.

Fourth, both in the theory and practice of control, the starting point is some predetermined goals. In other words, for predicting the evolution of complex economic systems, we have to assign plans that are in essence are rather proactive than corrective.

Fifth, deterministic methods are often used in formal modeling of a particular economic system. With such an approach, certainty is introduced into those situations where it does not actually exist. The inaccuracy of setting parameters during calculations is neglected, or under certain assumptions, inaccurate parameters are replaced by expert appraisals or average values. The resulting violations of equalities, balance relations, etc. make it necessary to vary some parameters for precisely satisfying the given conditions and obtaining an acceptable output. Such situations may occur due to insufficient knowledge of objects and also because of a person or group of persons participating in the control process. The peculiarity of such systems is that a significant part of the information required for their mathematical description exists in the form of beliefs or recommendations of experts.

1.2. **Notion of uncertainty and classification of uncertainty in economic systems.** The incomplete and/or inaccurate information on the conditions of implementing a chosen strategy is its inherent uncertainty. Uncertainty is caused by embarras du choix. For an economic system, the concept of uncertainty characterizes a situation in which there is no reliable information about the possible conditions of the internal and external environment, completely or partially. For example, V.V.

Cherkasov [4] considered uncertainty to be an incomplete or inaccurate representation of the values of various parameters in the future, caused by various reasons and, above all, incomplete or inaccurate information on the conditions of implementing decision, including costs and results.

Information about the external factors of an economic system is never absolutely sufficient, at least because it comes from the past and the present whereas a desired behavior of the system is oriented towards the future. The smaller the completeness and accuracy of information is and the longer the period for which the behavior of the system is planned, the greater the uncertainty will be.

F. Knight [5] understood a situation of uncertainty as a lack of awareness and the need to act based on opinion rather than knowledge.

Cherkasov interpreted uncertainty as the continuous variability of conditions, a fast and flexible reconfiguration of production, the actions of competitors, market changes, etc. He called uncertainty a most typical cause of risk in management.

There exist various approaches to classify the types of uncertainty. In the roughest classification, two classes are distinguished, namely, «good» uncertainty (some statistical or probabilistic characteristics for unknown factors are available) and «bad» uncertainty (such characteristics cannot be obtained in principle). Note that both types of uncertainty arising in real problems are taken into account using appropriate methods; for example, see [1].

In [6], the following classification of uncertainties was suggested:

- by degree of uncertainty: probabilistic, linguistic, interval, and complete uncertainty;
- by the nature of uncertainty: is parametric, structural, situational, and strategic uncertainty;
- by the use of information acquired during control: eliminable and ineliminable uncertainty.

V. S. Diev [7] presented more detailed classifications of uncertainties in modern economic systems.

1.3. **Sources of uncertainty in economic systems.** Considering the sources of uncertainty, we will distinguish three interconnected factors that cause uncertainty in economic systems [8].

1. The complexity factor: as a rule, an economic system is a large system that cannot be assigned a complete formal description, as well as a system with a variable

structure, a nontrivial hierarchy and internal contradictions that is often controlled using fuzzy criteria.

2. The human factor: human participation is an essential element that determines the behavior of an economic system at different levels and also affects various aspects of its operation. Moreover, the human factor manifests itself in the fact that many concepts, characteristics and parameters of economic behavior are formulated in natural language without an exact formal equivalent, which creates considerable (sometimes insurmountable) difficulties in modeling.
3. The external environment factor: for any economic system, the influence of other (external) systems has to be taken into account, which are often in conflict with the former.

In view of the above factors causing uncertainty in economic systems, we will divide the sources of uncertainty into three groups as follows.

1. Insufficient information about an economic system itself and about the processes running within it. Consequently, full-edged conclusions or assumptions on the evolution of an economic system and the final results cannot be made. In turn, such a situation may be due to
   - few data and other reasons that can be partially eliminated by organizing a system of timely and complete information support (for example, in technical systems, state monitoring is performed using information-measuring systems with inevitable errors, and the number of monitored parameters is limited, which do not prevent the appearance of some uncontrolled technical conditions, possibly causing disasters; in economic systems, the set of possible outcomes is well known, but the probability of a particular outcome can be unknown);
   - imperfect tools used to study an economic system, modeling errors, computational complexity, etc.
2. Accidental or deliberate counteraction of other economic agents. Such counteraction may have the form of violated contractual obligations by suppliers, uncertain demand for products, difficulties in marketing, or the behavior of local and regional authorities, both official and criminal. In addition, there are uncertainties caused by the competitive environment predetermining to a large extent the fate of a particular enterprise (e.g., industrial espionage, the penetration of competitors into trade secrets, and other effects on the internal affairs of a given enterprise).
3. The effect of random external factors that cannot be predicted due to their unexpectedness. Also, the impossibility of predicting further evolution of processes

due to the objectively inaccurate and ambiguous knowledge of the environment at the modern stage of science development. In particular,

– the uncertainties caused by insufficient knowledge of nature (e.g., the exact composition of supply of fish for a given fishing area in a given season is unknown);
– the uncertainties of the natural phenomena themselves (meteorological conditions affecting the average catch of fish, the mobility of supply of fish, etc.).

Thus, uncertainty is associated either with an insufficient amount of necessary information, or with the objective impossibility to acquire it and suggest reliable scenarios for the evolution of economic processes. In any case, the degree of uncertainty is determined by information, its amount, quality and timeliness.

2. Maximin in static case

This subsection is devoted to the single-criterion choice problem under uncertainty, which is described by an ordered triplet $\Gamma_1 = \langle X, Y, f(x,y)\rangle$.

Here the choice of a strategy (alternative) x from a set $X \subseteq \mathbb{R}^n$ is in charge of a decision-maker (DM). In economic systems, the role of DMs belongs to the general managers of industrial enterprises and business companies, the heads of states, sellers (suppliers) and buyers (customers); in mechanical control systems, to the captains of ships or aircrafts and the chiefs of control centers. In other words, a DM has right or authority to make decisions, give instructions and control their implementation. Each DM chooses from a given set of admissible actions, which will be called strategies. More specifically, a strategy is comprehended as a rule that associates with each state of the player's awareness a certain action (behavior) from a set of admissible actions (behaviors) given this awareness. Consider the case in which the DM's admissible strategies are the elements x of a well-defined set X. For a seller, a strategy is the price of one good; for the general manager of an industrial enterprise, strategies are production output, the amount of raw materials and equipment purchased, investments, innovations and implementation of new technologies, wages reallocation, penalties, bonuses, and other incentive and punishment mechanisms; for the captain of a ship, a strategy is own course (rudder angle, the direction and magnitude of reactive force).

In the single-criterion choice problem under uncertainty Γ_1, the DM's goal is to choose an appropriate strategy $x \in X$ for maximizing the values of a scalar criterion $f(x,y)$ (outcomes). The DM has to consider a possible realization of any uncertainty $y \in Y \subseteq \mathbb{R}^m$ within given limits. The value of $f(x,y)$ may indicate profit or production output. If the criterion $f_1(x, y)$ is associated with total losses or production cost (to be minimized), then the problem Γ_1 should be solved with $f(x, y) = -f_1(x, y)$, since

$$\max_{x \in X} f(x, y) = -\min_{x \in X} f_1(x, y).$$

Now, we proceed to uncertainty. The following situation seems common for almost everybody: it is necessary to reach a place of employment from home. First of all, a person in such conditions (further called passenger) has to decide which means of transportation to use (subway, bus, tramcar, suburban electric train, etc.). Choosing any means of transportation (strategy), passenger inevitably encounters incomplete and/or inaccurate information: delays or breakdowns of vehicles, sudden changes of schedule, strikes of drivers, weather fluctuations, crashes on routes, and other uncertainties. As was noted by O. Holmes, «The longing for certainty. . . is in every human mind. But certainty is generally illusion.»[2] At best passenger knows the variation ranges of these factors, without any probabilistic appraisals. Nevertheless, he/she has to make decision anyway! As a matter of fact, the incomplete and/or inaccurate information about the conditions under which his strategy is implemented makes its inherent uncertainty. In the problem Γ_1 denote by y a numerical value of uncertainty and by Y the set of all such values. We assume that the set Y is a priori given and non-empty.

Hereinafter, in accordance with the subject matter of this article, the n-dimensional vector x will be called the DM's strategy in the problem Γ_1 and $f(x, y)$ will be called his payoff function; the value of $f(x, y)$ for a specific pair $(x, y) \in X \times Y$ will be called an outcome for the strategy $x \in X$ and uncertainty $y \in Y$.

Interestingly, Γ_1 can be interpreted as a one-player game with nature.

First, we will introduce the concept of a guaranteed solution in outcomes of the problem Γ_1 and also its hierarchical interpretation using a two-level hierarchical game in the case where the interval uncertainty $y \in Y$ in Γ_1 is replaced by the strategic uncertainty $y(x) : X \to Y$, $y(\cdot) \in Y^X$.

2.1. **Formalization of guaranteed solution in outcomes.** The first attempt to solve the problem Γ_1 was undertaken by Wald in 1939; see [9]. It was based on the maximin principle, also known as the principle of guaranteed result. Let us formulate this principle in the following way.

[2]Oliver Wendell Holmes, Jr., byname The Great Dissenter, (1841–1935), was a justice of the United States Supreme Court, U.S. legal historian and philosopher who advocated judicial restraint.

Definition 1. The guaranteed solution in payoffs (outcomes) of the problem Γ_1 is a pair $(x^g, f^g) \in X \times \mathbb{R}$ determined by the chain of equalities

$$f^g = \max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} f(x^g, y). \tag{1}$$

The strategy x^g is called guaranteeing, and the value f^g the guaranteed outcome.

The whole essence of this solution can be explained as follows: choosing and using a strategy x^g, the DM guarantees an outcome f^g under any uncertainty $y \in Y$, since $f^g = \min_{y\in Y} f(x^g, y)$ implies

$$f(x^g, y) \geq f^g \qquad \forall y \in Y.$$

The maximin (1) includes two successive operations, namely, *first, the inner minimum*, which is intended to find an m-dimensional vector function $y(\cdot) : X \to Y$ such that, for each $x \in X$,

$$f(x, y(x)) = \min_{y\in Y} f(x, y),$$

and hence for each $x \in X$ it follows that

$$f(x, y) \geq f(x, y(x)) \qquad \forall y \in Y; \tag{2}$$

second, the outer maximum, which is intended to construct a strategy x^g such that

$$\max_{x\in X} f(x, y(x)) = f(x^g, y(x^g)) = f^g,$$

and hence

$$f^g = f(x^g, y(x^g)) \geq f(x, y(x)) \qquad \forall x \in X. \tag{3}$$

In fact, formula (3) means that among all minima of $f(x, y(x))$ in (2) for different $x \in X$, we choose the value f^g maximizing $f(x, y(x))$ in x, which is implemented on the strategy x^g.

Remark 1. Recall that we consider a special class of uncertainties of the form Y^X, which consists of the functions $y(x)$ with the domain X and the codomain Y. (The latter set is yielded by the inner minimum (2)). The actions of uncertainty are treated as the behavior of another (dummy) player, which has no payoff function and directs every effort to do as much harm to the DM as possible. (This is a strategic uncertainty in the terminology of Yu. B. Germeier.) The dummy player can use «any conceivable information. In particular, he/she possibly knows the DM's strategy.» [12, p. 353]. In this case, the so-called informational discrimination of the DM takes place [12, p. 353].

The inner minimum in (2) leads to a parametric problem: for each $x \in X$, find an m-dimensional vector function $y(\cdot) \in Y^X$ such that

$$\min_{y \in Y} f(x, y) = f(x, y(x)). \tag{4}$$

In this case, the following result should be taken into account.

Proposition 1. ([10, pp. 17–18]; [11, p. 54]) Let a scalar function $f(x, y)$ be continuous on $X \times Y$ and also let the sets X and Y be compact. Then

a) the function

$$\min_{y \in Y} f(x, y) = f(x, y(x)) \tag{5}$$

is continuous on X, and the multivalued mapping

$$Y(x) = \{y^* \in Y | f(x, y^*) = \min_{y \in Y} f(x, y)\} \qquad \forall x \in X,$$

i.e., $Y(x) : X \to Y$; has a Borel measurable selector $y(x)$.

b) Moreover, if $f(x, y)$ is strictly convex in $y \in Y$ for each $x \in X$ (i.e., for any $y^{(1)}, y^{(2)} \in Y$, $y^{(1)} \neq y^{(2)}$, and for each $x \in X$, the inequality

$$f(x, \lambda y^{(1)} + (1 - \lambda) y^{(2)}) < \lambda f(x, y^{(1)}) + (1 - \lambda) f(x, y^{(2)})$$

holds for any constants $\lambda \in (0; 1)$) and the set Y is convex, then the vector function $y(x)$ (5) is continuous on X.

Corollary 1. *If a scalar function $f(x, y)$ is continuous on $X \times Y$ and the sets X and Y are compact, then the function*

$$\max_{x \in X} f(x, y) \tag{6}$$

is continuous on Y (because $\min_{x \in X} [-f(x, y)] = -\max_{x \in X} f(x, y)$).

The following concepts are well known in game theory and will be used in further presentation: a) the strategy x^g defined by (1) is called the maximin strategy, and f^g is called the maximin; by analogy, the uncertainty y^0 from

$$f^0 = \min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} f(x, y^0)$$

is called the minimax uncertainty, and the value f^0 is called the minimax in the problem Γ_1; b) a pair $(x^g, y^0) \in X \times Y$ is a saddle point in the problem Γ_1 if

$$\max_{x \in X} f(x, y^0) = f(x^g, y^0) = \min_{y \in Y} f(x^g, y), \tag{7}$$

or equivalently,

$$\min_{y\in Y}\max_{x\in X} f(x,y) = f(x^g, y^0) = \max_{x\in X}\min_{y\in Y} f(x,y),$$

where x^g is the maximin strategy and y^0 is the minimax uncertainty in the problem Γ_1.

Proposition 2. Assume that in the problem Γ_1 the sets X and Y are compact and the function $f(x,y)$ is continuous on $X \times Y$. Then this problem has a guaranteed solution in outcomes (payoffs).

Proof. By Proposition 1 the function $\min\limits_{y\in Y} f(x,y)$ is continuous in $x \in X$ on the compact set X. According to the Weierstrass extreme-value theorem, a continuous function on a compact set X achieves maximum. □

2.2. **Interpretation of maximin within two-level hierarchical game.** Consider the following two-player game with a fixed sequence of moves. Assume player 1 (DM) is given priority in actions over player 2. Such a statement with the first move of player 1 describes well, e.g., an interaction of conflicting parties in two-level hierarchical systems. We will also accept the hypothesis that, whenever the outcome depends on the choice of player 2 only, he/she always minimizes the payoff function $f(x,y)$. Player 1 is informed about this behavior.

Then player 1 takes advantage of the first move, reporting his strategy $x \in X$ to player 2. Making the second move in this game, player 2 responds with a counter strategy $y(x) : X \to Y$ that minimizes the function $f(x, y(x))$ for each $x \in X$. If for each x this minimum is achieved at a unique point $y(x)$, then the best (guaranteed) result of player 1 makes up

$$f^g = \max_{x\in X}\min_{y\in Y} f(x,y) = \max_{x\in X} f(x, y(x)) = f(x^g, y(x^g)) = \min_{y\in Y} f(x^g, y).$$

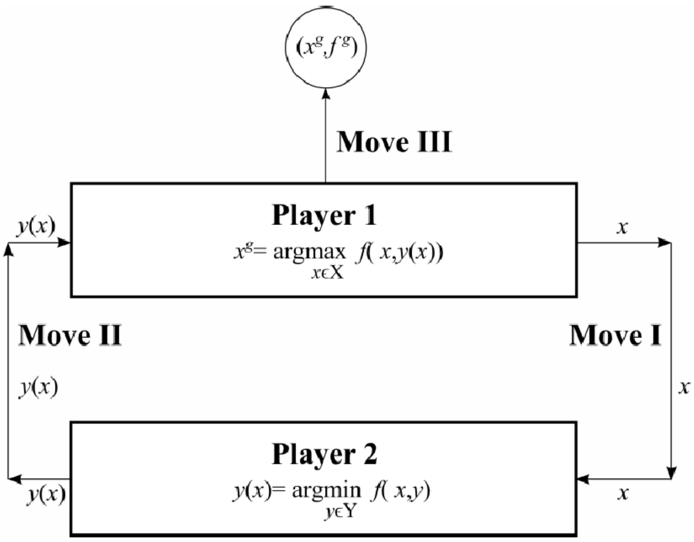
The sequence of moves of the DM and player 2 is illustrated in figure:

As a result, the DM prefers the maximin strategy x^g, which yields the guaranteed payoff

$$f^g \le f(x^g, y) \qquad \forall y \in Y.$$

Note that, for all $x \in X$, this payoff exceeds all other guaranteed payoffs:

$$\min_{y\in Y} f(x,y) \le f^g \qquad \forall x \in X.$$



3. Multistage maximin

For the difference statement of the linear-quadratic problem from section 2, the guaranteed solution in outcomes (maximin) is constructed using an appropriate modification of dynamic programming.

As the mathematical model, let us consider the ordered quadruple

$$\langle \mathbf{\Sigma}, \mathfrak{A}, \mathscr{Z}_u, \mathscr{J}(U, Z_u, x_0) \rangle, \tag{8}$$

which will be called the K-stage positional single-criterion linearquadratic problem under uncertainty. We make several assumptions regarding (8) as follows.

- The controlled system $\mathbf{\Sigma}$ evolves over time in accordance with the vector linear difference equation

$$x(k+1) = Ax(k) + u + z = f(k, x(k), u, z), \qquad x(0) = x_0, \tag{9}$$

  with the following notations: $k = 0, 1, \ldots, K{-}1$ as time instants, i.e., partition points of an entire time interval $[0, K]$ on which the controlled discrete process $\mathbf{\Sigma}$ is evolving, $x(k) \in \mathbb{R}^n$ as the value of the state vector x at a time instant $t = k$; $u \in \mathbb{R}^n$ as a DM's control action; $z \in \mathbb{R}^n$ as an uncertain factor, $(k, x(k))$ as a pair determining the position of (8) at a time instant k; $(0, x_0)$ as an initial position; A as a constant matrix of dimensions $n \times n$.
- A DM's positional strategy $U(k)$ at a time instant k is identified with a vector function $u(k, x) = P(k)x$, where $P(k) \in \mathbb{R}^{n\times n}$ is a constant matrix of dimensions $n \times n$. (This fact will be indicated by $U(k) \div u(k, x) = P(k)x$.) Hence, at a

time instant k an appropriate strategy is assigned by choosing a specific matrix $P(k) \in \mathbb{R}^{n\times n}$ of dimensions $n \times n$.

Thus, an ordered collection

$$U = (U(0), U(1), \dots, U(K-1)) \div$$
$$\div (u(0,x), u(1,x), \dots, u(K-1,x)) = (P(0)x, P(1)x, \dots, P(K-1)x)$$

is a DM's strategy in the problem (8), the set of all such strategies U will be denoted by $\mathfrak{A}$.

– The set of strategic positional uncertainties $Z_u(k)$ at a time instant k will be denoted by $\mathscr{Z}_u(k)$. It consists of

$$Z_u(k) \div z(k,x,u) = Q(k)x + R(k)u,$$

where $Q(k), R(k) \in \mathbb{R}^{n\times n}$ are constant matrices of specified dimensions. The special class of uncertainties that depend on the position (k,x) and also on the control action u has been selected due to the reasons discussed in Remark 1. As a result, the uncertainty in the problem (8) is described by the ordered collection

$$Z_u = (Z_u(0), Z_u(1), \dots, Z_u(K-1)) \div$$
$$\div (z(0,x,u), z(1,x,u), \dots, z(K-1,x,u)) =$$
$$= (Q(0)x + R(0)u, Q(1)x + R(1)u, \dots, Q(K-1)x + R(K-1)u);$$

the set of such uncertainties is denoted by $\mathscr{Z}_u$.

The controlled process in the problem (8) has the following dynamics over time. Assume that the DM has chosen and adopted a specific strategy $U \in \mathfrak{A}$:

$$U \div (u(0,x), u(1,x), \dots, u(K-1,x)) = (P(0)x, P(1)x, \dots, P(K-1)x).$$

Also, let some uncertainty $Z_u \in \mathscr{Z}_u$ have been realized in $\mathbf{\Sigma}$ regardless of this choice:

$$Z_u = (Z_u(0), \dots, Z_u(K-1)) \div (z(0,x,u), \dots, z(K-1,x,u)) =$$
$$= (Q(0)x + R(0)u, \dots, Q(K-1)x + R(K-1)u).$$

Substituting the above strategy U and uncertainty Z_u into (9), we obtain

$$x(1) = Ax_0 + u(0,x_0) + z(0,x_0,u(0,x_0)) =$$
$$= [A + P(0) + Q(0) + R(0)P(0)]x_0 =$$
$$= f(0,x_0,u(0,x_0),z(0,x_0,u(0,x_0))),$$
$$x(2) = [A + P(1) + Q(1) + R(1)P(1)]x(1) =$$

$$= f(1, x(1), u(1, x(1)), z(1, x(1), u(1, x(1)))),$$

$$\vdots$$

$$x(K) = [A + P(K-1) + Q(K-1) + R(K-1)P(K-1)]x(K-1) =$$
$$= f(K-1, x(K-1), u(K-1, x(K-1)),$$
$$z(K-1, x(K-1), u(K-1, x(K-1)))).$$

This gives three sequences,

$$\{x(k)\}_{k=0}^{K},$$
$$\{u[k] = P(k)x(k)\}_{k=0}^{K},$$
$$\{z[k] = Q(k)x(k) + R(k)P(k)x(k)\}_{k=0}^{K},$$

which form the criterion (also called the payoff or utility function of the DM)

$$\mathscr{J}(U, Z_u, x_0) = x'(K)Cx(K) + \\ + \sum_{k=0}^{K-1}(u'[k]D(k)u[k] + z'[k]L(k)z[k]) = \\ = \Phi(x(K)) + \sum_{k=0}^{K-1} F(k, x(k, u[k], z[k])). \tag{10}$$

A value of the function (10) is called an outcome or DM's payoff. In formula (10), all matrices C, $D(k)$, $L(k)$, $P(k)$, $Q(k)$ and $R(k)$ of dimensions $n \times n$ are constant, and the matrices C, $D(k)$ and $L(k)$ are symmetric. Recall that the prime indicates transposition, for a symmetric matrix $M \in \mathbb{R}^{n\times n}$, the expression $M > 0$ (< 0) shows that the quadratic form $u'Mu$ is positive (negative, respectively) definite; E_n is an identity matrix of dimensions $n \times n$; 0_n is a zero n-dimensional vector; finally, $Idem\{u \to u^g\}$ means the bracketed expression with u replaced by u^g. In addition, $detB$ denotes the determinant of a square matrix B.

At conceptual level, choosing his strategy $U \in \mathfrak{A}$, the DM seeks to maximize the outcome $\mathscr{J}(U, Z_u, x_0)$ in the problem (8) under any realization of the uncertainty $Z_u \in \mathscr{Z}_u$. Definition 1 naturally leads to the following concept.

A pair $(U^g, \mathscr{J}^g[x_0]) \in \mathfrak{A} \times \mathbb{R}$ will be called the guaranteed solution in outcomes of the problem (8) if there exists an uncertainty $Z_u^g \in \mathscr{Z}_u$ such that

$$\mathscr{J}^g[x_0] = \max_{U\in\mathfrak{A}} \min_{Z_u\in\mathscr{Z}_u} \mathscr{J}(U, Z_u, x_0) = \\ = \min_{Z_u\in\mathscr{Z}_u} \mathscr{J}(U^g, Z_u, x_0) = \mathscr{J}(U^g, Z_u^g, x_0). \tag{11}$$

In this case, U^g will be called the guaranteeing strategy, and $\mathscr{J}^g[x_0]$ the guaranteed outcome.

Remark 2. The concept of guaranteed solution in outcomes suggests the DM to use the strategy $U^g \in \mathfrak{A}$ in the problem (8) on two grounds as follows. First, the equality

$$\min_{Z_u \in \mathscr{Z}_u} \mathscr{J}(U^g, Z_u, x_0) = \mathscr{J}^g[x_0]$$

implies that $\mathscr{J}(U^g, Z_u, x_0) \geq \mathscr{J}^g[x_0]$ under any uncertainty realization of the uncertainty $Z_u \in \mathscr{Z}_u$. In other words, with this strategy the outcome will be not smaller than the guaranteed outcome $\mathscr{J}^g[x_0]$ (the lower bound on $\mathscr{J}(U^g, Z_u, x_0)$ over all $Z_u \in \mathscr{Z}_u$).

Second, for each strategy $U \in \mathfrak{A}$ the DM will obtain the guaranteed outcome $\min\limits_{Z_u \in \mathscr{Z}_u} \mathscr{J}(U, Z_u, x_0)$, which is not greater than $\mathscr{J}^g[x_0]$.

Now, we introduce sufficient conditions for the existence of the multistage maximin (11) that are based on dynamic programming. At each time instant $k = K, K–1, K–2, \ldots, 1, 0$, we will use the Bellman function

$$V^{(k)}(x) = x'\Theta(k)x,$$

with a symmetric matrix $\Theta(k) \in \mathbb{R}^{n\times n}$ as well as scalar functions

$$\begin{gathered} W(k, x, u, z, V^{(k+1)}(Ax + u + z)) = \\ = W(k, x, u, z, (x'A' + u' + z')\Theta(k + 1)(Ax + u + z)) = \\ = W[k, x, u, z, \Theta(k + 1)] = u'D(k)u + z'L(k)z + \\ +(x'A' + u' + z')\Theta(k + 1)(Ax + u + z) \\ (k = K - 1, K - 2, \ldots, 1, 0). \end{gathered} \tag{12}$$

Proposition 3. Let $\{V^{(k)}(x) = x'\Theta(k)x\}_{k=0}^{K-1}$, $\{u(k, x, \Theta(k+1)) = P(k, \Theta(k+1))x\}\}_{k=0}^{K-1}$ and $\{z(k, x, u, \Theta(k + 1)) = Q(k, \Theta(k + 1))x + R(k, \Theta(k + 1))u\}_{k=0}^{K-1}$ be three sequences, the first composed of scalar functions and the last two of n-dimensional vector functions, that satisfy the following assumptions:

$$V^{(K)}(x) = x'Cx \qquad \forall x \in \mathbb{R}^n, \tag{13}$$

for all x, $u \in \mathbb{R}^n$, $\Theta(k + 1) \in \mathbb{R}^{n\times n}$, and $k = K - 1, \ldots, 1, 0$,

$$\begin{gathered} \min_z W[k, x, u, z, \Theta(k + 1)] = \\ = W[k, x, u, z(k, x, u, \Theta(k + 1)), \Theta(k + 1)]; \end{gathered} \tag{14}$$

for each $x \in \mathbb{R}^n$, $\Theta(k+1) \in \mathbb{R}^{n\times n}$ and $k = K-1, \dots, 1, 0$,

$$\max_u W[k, x, u, z(k, x, u, \Theta(k+1)), \Theta(k+1)] = \\ = W[k, x, u(k, x, \Theta(k+1)), \\ z(k, x, u(k, x, \Theta(k+1)), \Theta(k+1)), \Theta(k+1)]; \tag{15}$$

for any $x \in \mathbb{R}^n$ and $k = K-1, K-2, \dots, 1, 0$,

$$V^{(k)}(x) = x'\Theta(k)x = W[k, x, u(k, x, \Theta(k+1)), \\ z(k, x, u(k, x, \Theta(k+1)), \Theta(k+1)), \Theta(k+1)]. \tag{16}$$

Then for any initial state vector $x_0 \in \mathbb{R}^n$, the guaranteed solution in outcomes $(U^g, \mathscr{J}^g[x_0])$ of the problem (8) has the following form:
the guaranteed strategy is given by

$$U^g \div (u^g[0, x], u^g[1, x], \dots, u^g[K-1, x]),$$

where

$$u^g[k, x] = u(k, x, \Theta(k+1)) \qquad (k = 0, 1, \dots, K-1),$$

and the guaranteed outcome is given by

$$\mathscr{J}^g[x_0] = V^{(0)}(x_0) = x_0'\Theta(0)x_0.$$

(The notations are the same as in (12). Formula (16) is used to successively find the matrices $\Theta(k)$ $(k = K-1, \dots, 1, 0)$.)

Proof. This result can be established by a standard procedure, for example, see [13, pp. 366–367]. □

Remark 3. For each time instant $(k = 0, 1, \dots, K–1)$, consider the auxiliary problem

$$\Gamma(k) = \langle \mathfrak{A}, \mathscr{Z}, W[k, x, u, z, \Theta(k+1)] \rangle,$$

where $\mathfrak{A}$ is the set of strategies $U = (U(0), U(1), \dots, U(K-1)) \div (u(0, x), u(1, x), \dots$ $\dots, u(K-1, x))$ of the form $u(k, x) = P(k)x$; $\mathscr{Z}$ denotes the set of uncertainties $z(k, x, u, \Theta(k+1)) = Q(k)x + R(k)u$; the criterion $W[k, x, u, z, \Theta(k+1)]$ is given by (12). Then merging the requirements (14) and (15) actually means the equalities

$$\max_{u\in\mathfrak{A}} \min_{z\in\mathscr{Z}} W[k, x, u, z, \Theta(k+1)] = \\ = \min_{z\in\mathscr{Z}} W[k, x, u(k, x, \Theta(k+1)), z, \Theta(k+1)] = W[k, x, u(k, x, \Theta(k+1)), \\ z(k, x, u(k, x, \Theta(k+1)), \Theta(k+1)), \Theta(k+1)] = W^g[k, x, \Theta(k+1)]. \tag{17}$$

In other words, at each time instant $k = K-1, K-2, \ldots, 1, 0$ the DM implements the maximin (17) in the auxiliary problem $\Gamma(k)$. Consequently, according to Proposition 3, implementing the local maximin at each time instant $k = K-1, K-2, \ldots, 1, 0$, the DM actually arrives at the global maximin (11) in the problem (8).

Taking advantage of Proposition 3, we will find an explicit form of the guaranteed solution in outcomes of the problem (8). Before doing it, let us present three auxiliary results. Recall that if a quadratic form $z'Gz$ is positive (negative) definite and $G = G' \in \mathbf{R}^{n\times n}$, then all n roots λ_i of the characteristic equation $det[G–\lambda E_n] = 0$ are real and $\lambda_i > 0$ ($\lambda_i < 0$, respectively).

Lemma 1. *Consider symmetric matrices $L(k-1) > 0$ and $\Theta(k) < 0$ of dimensions $n\times n$. The inequality*

$$L(k-1) + \Theta(k) > 0$$

holds if $\lambda(k-1) > \mu(k)$, where $\lambda(k-1)$ and $-\mu(k)$ are the least roots of the characteristic equations $det[L(k-1) - \lambda E_n] = 0$ and $det[\Theta(k) - \mu E_n] = 0$, respectively.

Proof. Let $\lambda(k-1)$ and $-\mu(k)$ be the least roots of the corresponding characteristic equations. In this case,

$$z'L(k-1)z \geq \lambda(k-1)z'z, \qquad z'\Theta(k)z \geq -\mu(k)z'z \qquad \forall z \in \mathbb{R}^n,$$

and hence

$$z'[L(k-1) + \Theta(k)]z \geq [\lambda(k-1) - \mu(k)]z'z > 0 \qquad \forall z \in \mathbb{R}^n \setminus \{0_n\}.$$

□

Lemma 2. *Consider symmetric constant matrices $D(k)$, C, $L(k-1)$, and $\Theta(k)$ of dimensions $n\times n$ such that*

$$D(k) < 0, C < 0, L(k-1) > 0, L(k-1) + \Theta(k) > 0. \tag{18}$$

Then the matrices

$$M(\Theta(k)) = \Theta(k)\{\Theta^{-1}(k) - [L(k-1) + \Theta(k)]^{-1}\}\Theta(k),$$
$$\Theta(k-1) = A'M(\Theta(k)) \times \times\{M^{-1}(\Theta(k)) - [D(k) + M(\Theta(k))]^{-1}\}M(\Theta(k))A$$

are also symmetric, $M(\Theta(k)) < 0$, and $\Theta(k-1) < 0$ if

$$detA \neq 0. \tag{19}$$

Proof. The symmetry of the matrices $M(\Theta(k))$ and $\Theta(k-1)$ follows from the properties $(AB)' = B'A'$, $[A^{-1}]' = [A']^{-1}$, $A'' = A$ and the two easily checked equalities $M(\Theta(k)) = M'(\Theta(k))$ and $\Theta(k-1) = \Theta'(k-1)$.

The negative definiteness of $M(\Theta(k))$ and $\Theta(k-1)$ is established by the chain of implications

$$\Theta(k) < 0 \Rightarrow det\Theta(k) \neq 0 \Rightarrow \exists\Theta^{-1}(k) \wedge \Theta^{-1}(k) < 0,$$
$$[L(k-1) + \Theta(k)] > 0 \Rightarrow [L(k-1) + \Theta(k)]^{-1} > 0 \Rightarrow -[L(k-1) + \Theta(k)]^{-1} < 0,$$
$$\Theta^{-1}(k) < 0 \wedge -[L(k-1) + \Theta(k)]^{-1} < 0 \Rightarrow$$
$$\Rightarrow \Theta^{-1}(k) - [L(k-1) + \Theta(k)]^{-1} < 0 = \{det\Theta(k) \neq 0\} \Rightarrow$$
$$\Rightarrow M(\Theta(k)) = \Theta(k)\{\Theta^{-1}(k) - [L(k-1) + \Theta(k)]^{-1}\}\Theta(k) < 0;$$
$$D(k) < 0 \wedge M(\Theta(k)) < 0 \Rightarrow [D(k) + M(\Theta(k))] - M(\Theta(k)) = D(k) < 0 =$$
$$= \{\ [14,\text{ p. }89]\} \Rightarrow M^{-1}(\Theta(k)) - [D(k) + M(\Theta(k))]^{-1} < 0;$$
$$detA \neq 0 \wedge M(\Theta(k)) < 0 \Rightarrow det[AM(\Theta(k))] \neq 0,$$
$$det[AM(\Theta(k))] \neq 0 \wedge M^{-1}(\Theta(k)) - [D(k) + M(\Theta(k))]^{-1} < 0 \Rightarrow \Theta(k-1) =$$
$$= A'M(\Theta(k))\{M^{-1}(\Theta(k)) - [D(k) + M(\Theta(k))]^{-1}\}M(\Theta(k))A < 0.$$

□

Corollary 2. *Under conditions (18) and (19) (see Lemma 2), we have the implication*

$$[\Theta(k) < 0] \Rightarrow [\Theta(k-1) < 0]. \tag{20}$$

In fact, the validity of (20) has be demonstrated by Lemma 2.

Remark 4. The concept of guaranteed solution in outcomes itself directly leads to the following design method of the guaranteed solution of the problem (8) using Proposition 3, see the stages described below. The Bellman functions $V^{(k)}(x)$ have to be constructed as quadratic forms $V^{(k)}(x) = x'\Theta(k)x$, with symmetric matrices $\Theta(k) \in \mathbb{R}^{n\times n}$.

Stage 1 ($k = K$). From (9), due to

$$V^{(K)}(x) = x'\Theta(K)x = x'Cx \qquad \forall x \in \mathbb{R}^n,$$

find the matrix $\Theta(K) = C$.

Stage 2 ($k = K$–1). The function $W[K-1, x, u, z, \Theta(K)]$ (12) takes the form

$$W[K-1, x, u, z, \Theta(K)] = u'D(K-1)u + z'L(K-1)z+$$
$$+(x'A' + u' + z')\Theta(K)(Ax + u + z). \tag{21}$$

Checking the condition $L(K-1)+C = L(K-1)+\Theta(K) > 0$, construct $z(K-1, x, u, \Theta(K))$ in accordance with

$$\min_z W[K-1, x, u, z, \Theta(K)] =$$
$$= W[K-1, x, u, z(K-1, x, u, \Theta(K)), \Theta(K)] \qquad \forall x, u \in \mathbb{R}^n. \tag{22}$$

Next, calculate the vector function $u(K-1, x, \Theta(k))$ in accordance with

$$\max_u W[K-1, x, u, z(K-1, x, u, \Theta(K)), \Theta(K)] =$$
$$= W[K-1, x, u(K-1, x, \Theta(K)),$$
$$z(K-1, x, u(K-1, x, \Theta(K)), \Theta(K)), \Theta(K)] == \overline{W}[K-1, x] \qquad \forall x \in \mathbb{R}^n,$$

and find the constant matrix $\Theta(K-1)$ of dimensions $n \times n$ from the identity

$$x'\Theta(K-1)x = \overline{W}[K-1, x] \qquad \forall x \in \mathbb{R}^n. \tag{23}$$

Thus, Stage 2 yields the n-dimensional vector function

$$u^g[K-1, x] = u(K-1, x, \Theta(K) = C) = P(K-1, \Theta(K))x$$

and also the symmetric matrix $\Theta(K-1) \in \mathbb{R}^{n\times n}$.

Then, repeating all operations of Stage 2 for $k = K–2$, obtain the vector function $u^g[K-2, x] = u(K-2, x, \Theta(K-1)) = P(K-2, \Theta(K-1))x$ and the matrix $\Theta(K-2) \in \mathbb{R}^{n\times n}$ of dimensions $n \times n$. And so on, for $k = K-3, \dots, 1$.

Finally, repeat the operations of Stage 2 for $k = 0$, replacing $\Theta(K)$ by $\Theta(1)$. For $k = 0$,

$$W[0, x, u, z, \Theta(1)] = u'D(0)u + z'L(0)z+$$
$$+(x'A' + u' + z')\Theta(1)(Ax + u + z).$$

Check the requirement $L(0) + \Theta(1) > 0$ and construct the vector function $z(0, x, u, \Theta(1))$ in accordance with

$$\min_z W[0, x, u, z, \Theta(1)] =$$
$$= W[0, x, u, z(0, x, u, \Theta(1)), \Theta(1)] \qquad \forall x, u \in \mathbb{R}^n.$$

Next, find the n-dimensional vector function $u(0, x, \Theta(1))$ in accordance with

$$\max_u W[0, x, u, z(0, x, u, \Theta(1)), \Theta(1)] =$$
$$= W[0, x, u(0, x, \Theta(1)),$$
$$z(0, x, u(0, x, \Theta(1)), \Theta(1)), \Theta(1)] = \overline{W}[0, x] \qquad \forall x \in \mathbb{R}^n,$$

and also the matrix $\Theta(0)$ of dimensions $n \times n$ from the identity

$$x'\Theta(0)x = \overline{W}[0,x] \qquad \forall x \in \mathbb{R}^n.$$

As a result, the vector function $u^g[0,x] = u(0,x,\Theta(1)) = P(0,\Theta(1))x$ and the constant matrix $\Theta(0)$ of dimensions $n \times n$ are obtained.

Thus, for any initial state vector $x(0) = x_0 \neq 0_n$ in (9), the guaranteed solution in outcomes $(U^g, \mathscr{J}^g[x_0])$ of the problem (8) has the explicit form

$$\begin{gathered} U^g \div (u^g[0,x], u^g[1,x], \dots, u^g[K-1,x]), \\ u^g[k,x] = u(k,x,\Theta(k+1)) = P(k,\Theta(k+1))x \qquad (k = 0,1,\dots,K-1), \\ \mathscr{J}^g[x_0] = x_0'\Theta(0)x_0. \end{gathered} \tag{24}$$

In view of the stages described in Remark 4, we may formulate the following result.

Proposition 4. Consider the problem (8) with

$$C < 0, \quad det A \neq 0, \quad D(k) < 0, \quad L(k) > 0 \qquad (k = 0,1,\dots,K-1) \tag{25}$$

and let the sequence of matrices $\{\Theta(k)\}_{k=0}^{K}$ constructed by the recursive formulas

$$\begin{gathered} \Theta(K) = C, \\ M(\Theta(K)) = C[C^{-1} - (L(K-1)+C)^{-1}]C, \\ \Theta(K-1) = A'M(\Theta(K))\{M^{-1}(\Theta(K))- \\ -[D(K-1)+M(\Theta(K))]^{-1}\}M(\Theta(K))A, \\ M(\Theta(K-1)) = \Theta(K-1)[\Theta^{-1}(K-1)- \\ -(L(K-2)+\Theta(K-1))^{-1}]\Theta(K-1), \\ \Theta(K-2) = A'M(\Theta(K-1))\{M^{-1}(\Theta(K-1))- \\ -[D(K-2)+M(\Theta(K-1))]^{-1}\}M(\Theta(K-1))A, \\ \vdots \\ \Theta(k) = A'M(\Theta(k+1))\{M^{-1}(\Theta(k+1))- \\ -[D(k)+M(\Theta(k+1))]^{-1}\}M(\Theta(k+1))A, \\ M(\Theta(k)) = \Theta(k)[\Theta^{-1}(k) - (L(k-1)+\Theta(k))^{-1}]\Theta(k), \\ \Theta(k-1) = A'M(\Theta(k))\{M^{-1}(\Theta(k)) - [D(K-1)+M(\Theta(k))]^{-1}\}M(\Theta(k))A, \\ \vdots \\ \Theta(1) = A'M(\Theta(2))\{M^{-1}(\Theta(2)) - [D(1)+M(\Theta(2))]^{-1}\}M(\Theta(2))A, \end{gathered}$$

$$M(\Theta(1)) = \Theta(1)[\Theta^{-1}(1) - (L(0) + \Theta(1))^{-1}]\Theta(1),$$
$$\Theta(0) = A'M(\Theta(1))\{M^{-1}(\Theta(1)) - [D(0) + M(\Theta(1))]^{-1}\}M(\Theta(1))A \qquad (26)$$

be such that

$$L(k-1) + \Theta(k) > 0 \qquad (k = K, K-1, \ldots, 1). \qquad (27)$$

Then for any initial state vector $x_0 \in \mathbb{R}^n$ in equation (9), the guaranteed solution in outcomes $(U^g, \mathscr{J}^g[x_0])$ of the problem (8) has the form

$$U^g \div (-[D(0) + M(\Theta(1))]^{-1}M(\Theta(1))Ax, \ldots$$
$$\ldots, -[D(K-1) + M(\Theta(K))]^{-1}M(\Theta(K))Ax), \qquad (28)$$
$$\mathscr{J}^g[x_0]) = x_0'\Theta(0)x_0.$$

Proof. In accordance with Stage 1, the matrix $\Theta(K)$ is $\Theta(K) = C < 0$ and the Bellman function at the time instant $k = K$ is given by

$$V^{(K)}(x) = x'\Theta(K)x = x'Cx.$$

Following the recommendations of Stage 2, we construct the scalar function (12) for $k = K$–1

$$W[K-1, x, u, z, \Theta(K)] = u'D(K-1)u + z'L(K-1)z+$$
$$+(x'A' + u' + z')\Theta(K)(Ax + u + z), \qquad (29)$$

and find $z(K-1, x, u, \Theta(K))$ from (14), i.e.,

$$\min_z W[K-1, x, u, z, \Theta(K)] = Idem[z \to z(K-1, x, u, \Theta(K))].$$

Due to the above explicit form of $W[K-1, x, u, z, \Theta(K)]$, the vector function $z(K-1, x, u, \Theta(K))$ simultaneously minimizes the function

$$\varphi_1(K-1, x, u, z) = z'L(K-1)z+$$
$$+z'\Theta(K)z + 2z'\Theta(K)(Ax + u) \qquad \forall x, u \in \mathbb{R}^n.$$

Here, the sufficient conditions can be written as

$$grad_z\varphi_1(K-1, x, u, z)|_{z(K-1,x,u,\Theta(K))} =$$
$$= \frac{\partial\varphi_1(K-1, x, u, z)}{\partial z}|_{z(K-1,x,u,\Theta(K))} =$$
$$= 2[L(K-1) + \Theta(K)]z(K-1, x, u, \Theta(K)) + 2\Theta(K)(Ax + u) = 0_n \qquad \forall x, u \in \mathbb{R}^n, \qquad (30)$$

and the Hessian has the form

$$\frac{\partial^2 \varphi_1(K-1,x,u,z)}{\partial z^2} = 2[L(K-1)+\Theta(K)] > 0.$$

The last inequality is immediate from (27) with $k = K$. On the other hand, condition (30) implies, first,

$$z(K-1,x,u,\Theta(K)) = -[L(K-1)+\Theta(K)]^{-1}\Theta(K)(Ax+u), \qquad (31)$$

and second,

$$z'(K-1,x,u,\Theta(K))[L(K-1)+\Theta(K)]z(K-1,x,u,\Theta(K))+$$
$$+2z'(K-1,x,u,\Theta(K))\Theta(K)(Ax+u) =$$
$$= -z'(K-1,x,u,\Theta(K))[L(K-1)+\Theta(K)]z(K-1,x,u,\Theta(K)).$$

Using this relation, equality (31) and the first row of formula (2) with $k = K$, we obtain the following chain of equalities from (29) with $z = z(K-1,x,u,\Theta(K))$:

$$W[K-1,x,u,z(K-1,x,u,\Theta(K)),\Theta(K)] =$$
$$= u'D(K-1)u + (x'A'+u')\Theta(K)(Ax+u)-$$
$$-z'(K-1,x,u,\Theta(K))[L(K-1)+\Theta(K)]z(K-1,x,u,\Theta(K)) =$$
$$= u'D(K-1)u + (x'A'+u')\Theta(K)\Theta^{-1}(K)\Theta(K)(Ax+u)-$$
$$-(x'A'+u')\Theta(K)[L(K-1)+\Theta(K)]^{-1}\Theta(K)(Ax+u) =$$
$$= u'D(K-1)u + (x'A'+u')\Theta(K)\{\Theta^{-1}(K)-$$
$$-[L(K-1)+\Theta(K)]^{-1}\}\Theta(K)(Ax+u) =$$
$$= u'D(K-1)u + (x'A'+u')M(\Theta(K))(Ax+u) =$$
$$= u'[D(K-1)+M(\Theta(K))]u+$$
$$+2u'M(\Theta(K))(Ax+u) + x'A'M(\Theta(K))Ax.$$

Now, we get back to (4), taking into account the formula

$$W[K-1,x,u,z(K-1,x,u,\Theta(K)),\Theta(K)] =$$
$$= u'D(K-1)u + (x'A'+u')M(\Theta(K))(Ax+u) =$$
$$= u'[D(K-1)+M(\Theta(K))]u+$$
$$+2u'M(\Theta(K))(Ax+u) + x'A'M(\Theta(K))Ax.$$

If the maximum in (4) is achieved at $u = u(K-1,x,\Theta(K))$, then

$$\max_u \varphi_2(K-1,x,u) =$$

$$= \max_u \{u'[D(K-1) + M(\Theta(K))]u + 2u'M(\Theta(K))Ax\} \qquad \forall x \in \mathbb{R}^n \tag{32}$$

is also implemented at the same $u = u(K-1, x, \Theta(K))$. The sufficient conditions of this maximum can be written as

$$\frac{\partial \varphi_2(K-1,x,u)}{\partial u}|_{u(K-1,x,\Theta(K))} =$$
$$= 2[D(K-1) + M(\Theta(K))]u(K-1,x,\Theta(K))+$$
$$+2M(\Theta(K))Ax = 0_n \qquad \forall x \in \mathbb{R}^n, \tag{33}$$
$$\frac{\partial^2 \varphi_1(K-1,x,u)}{\partial u^2} = 2[D(K-1) + M(\Theta(K))] < 0.$$

The second requirement is satisfied due to $D(K-1) < 0$ (see (24) with $k = K$–1) and $M(\Theta(K)) < 0$. In addition, the matrix $M(\Theta(K))$ has symmetry by Lemma 2 with $k = K$.

From (33) it follows that, first,

$$u(K-1,x,\Theta(K)) = -[D(K-1) + M(\Theta(K))]^{-1}M(\Theta(K))Ax; \tag{34}$$

second,

$$u'(K-1,x,\Theta(K))[D(K-1) + M(\Theta(K))]u(K-1,x,\Theta(K))+$$
$$+2u'(K-1,x,\Theta(K))M(\Theta(K))Ax = -u'(K-1,x,\Theta(K))[D(K-1)+$$
$$+M(\Theta(K))]u(K-1,x,\Theta(K)) \qquad \forall x \in \mathbb{R}^n.$$

In view of this identity, (34), and the second row of formula (2), we obtain

$$\overline{W}[K-1,x] = W[K-1,x,u(K-1,x,\Theta(K)),$$
$$z(K-1,x,u(K-1,x,\Theta(K)),\Theta(K)),\Theta(K)] =$$
$$= -u'(K-1,x,\Theta(K))[D(K-1)+$$
$$+M(\Theta(K))]u(K-1,x,\Theta(K))+$$
$$+x'A'M(\Theta(K))M{-}1(\Theta(K))M(\Theta(K))Ax = \tag{35}$$
$$= x'A'M(\Theta(K))\{M^{-1}(\Theta(K))-$$
$$-[D(K-1) + M(\Theta(K))]^{-1}\}M(\Theta(K))Ax =$$
$$= x'\Theta(K-1)x = V^{(K-1)}(x).$$

Moreover, by Corollary 2,

$$[\Theta(K)(=C) < 0] \Rightarrow [\Theta(K-1) < 0],$$

and by Lemma 2 the matrix $\Theta(K-1)$ is symmetric.

The same considerations can be applied to the case $k = K{-}2$ by simply replacing the matrix $\Theta(K)$ with $\Theta(K-1)$ and the number $K{-}1$ with $K{-}2$ in all formulas starting from (29). Following this approach, we establish the analogs of (34) and (35),

$$u(K-2, x, \Theta(K-1)) = -[D(K-2) + M(\Theta(K-1))]^{-1} M(\Theta(K-1))Ax,$$

and

$$V^{(K-2)}(x) = x'\Theta(K-2)x,$$

respectively, where the nonnegative definite matrices $M(\Theta(K-1))$ and $\Theta(K-2)$ are given by (19) with $k = K{-}1$.

Similar operations should be performed for $k = K-3, \ldots, 1, 0$. By mathematical induction on k, for each $k = 0, 1, \ldots, K{-}1$ we get

$$u(k, x, \Theta(k+1)) = -[D(k) + M(\Theta(k+1))]^{-1} M(\Theta(k+1))Ax, \quad V^{(k)}(x) = x'\Theta(k)x. \tag{36}$$

Finally, the end of Remark 4 in combination with the first and second rows of formula (36) with $k = 0, 1, \ldots, K-1$ and $k = 0$, respectively, allows us to prove (25). □

Remark 5. For obtaining the guaranteed solution in outcomes of the linear-quadratic discrete single-criterion problem (8)–(11) using Proposition 4, we have to first, check the constraints (25), second, for $k = K, K-1, \ldots$, construct the two sequences

$$\{\Theta(K), \Theta(K-1), \ldots, \Theta(1), \Theta(0)\},$$

and

$$\{M(\Theta(K)), M(\Theta(K-1)), \ldots, M(\Theta(1)), M(\Theta(0))\}$$

by the recursive relations (26); third, check whether the requirements (27) are satisfied; if so, analytically design the guaranteed solution in outcomes $(U^g, \mathcal{J}^g[x_0])$ by formulas (28).

Conclusion

The article consists of two parts. The first part is devoted to general questions that are related to uncertainty: causes and sources of uncertainties appearance, classification of uncertainties in economic systems and approach to their assessment. In the second part the concept of maximin, based on the principle of guaranteed result (Wald's principle) is considered. In this case, maximin is interpreted from viewpoint of two-level hierarchical game. On the basis of the maximin concept, a guaranteed solution in outcomes for K-stage positional single-criterion linear quadratic problem under uncertainty is formalized.

An explicit form of the guaranteed solution for this problem is found.The article opens the theoretical direction of the research of dynamic multi-stage positional games under uncertainty.

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