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ON PERIODIC SOLUTIONS OF LINEAR PARABOLIC PROBLEMS WITH NONLOCAL BOUNDARY CONDITIONS

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On periodic solutions of linear parabolic problems with nonlocal boundary conditions.

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Abstract. A linear parabolic equation with nonlocal boundary conditions of the Bitsadze-Samarsky type is considered. The existence and uniqueness theorem of the periodic solution is proved.

Keywords: *nonlocal problem, parabolic equation, monotone operator.*

Nonlocal elliptic boundary value problems have been considered since the 30s of the XXth century in the work of T. Carleman. In the 50–60s of the XXth century, the abstract nonlocal elliptic problems were studied by M. I. Vishik, F. Browder, etc. Nonlocal parabolic problems in a bounded cylinder were considered mainly in the cases of parabolic delay differential equations, parabolic integro–differential equations, and parabolic operator– differential equations. In this paper, we consider a parabolic equation with nonlocal boundary conditions of the Bitsadze-Samarsky type, cf. [1]. The peculiarity of these nonlocal conditions is that they are set using shifts in spatial variables in a bounded domain. A method for studying elliptic boundary value problems with such nonlocal conditions was developed in the 80-90s, see [2–4]. In this paper, the time–periodic solutions of a linear parabolic equation with nonlocal boundary conditions are investigated. The proofs are given for a model example. However the method is suitable for the general case of nonlocal boundary conditions of this type.

In the rectangular parallelepiped $\Omega_T = (0,T)\times(0,2)\times(0,1)$ we consider the parabolic equation

$$\partial_t w(t,x) - \sum_{i,j=1,2} \partial_i \left(A_{ij}(t,x)\partial_j w(t,x)\right) = f(t,x) \qquad ((t,x)\in\Omega_T), \tag{1}$$

with nonlocal boundary conditions

$$\begin{cases} w(t,x_1,0) = w(t,x_1,1) = 0 & (0 < t < T; 0 \le x_1 \le 2), \\ w(t,x)|_{x_1=0} = \gamma_1 w(t,x)|_{x_1=1}, & (0 < t < T; 0 < x_2 < 1), \\ w(t,x)|_{x_1=2} = \gamma_2 w(t,x)|_{x_1=1} & (0 < t < T; 0 < x_2 < 1). \end{cases} \tag{2}$$

Here $f \in L_2(\Omega_T)$, the functions $A_{ij} \in C^\infty(\mathbb{R}^3)$ are 1–periodic in x_1 and T–periodic in t. Moreover, $A_{ij}(t,x) = A_{ji}(t,x)$ $(i,j = 1,2)$, and there exists $c_1 > 0$ such that

$$\sum_{i,j=1,2} A_{ij}(t,x)\xi_i\xi_j \ge c_1 \sum_{i=1,2} |\xi_i|^2 \qquad \forall (t,x) \in \overline{\Omega_T}. \tag{3}$$

Time–periodic solution of (1)–(2) must satisfy the condition

$$w(0,x) = w(T,x) \qquad (x = (x_1,x_2) \in Q = (0,2) \times (0,1)). \tag{4}$$

We consider our problem in Sobolev space $L_2(0,T;W_2^1(Q))$, this is the set of functions $u \in L_2(\Omega_T)$ such that $\partial_i u \in L_2(\Omega_T)$. Let

$$L_2(0,T;W^1_{2,\gamma}(Q)) := \{w \in L_2(0,T;W_2^1(Q)) : w \,\text{satisfies (2)}\}. \tag{5}$$

In this paper we consider the spaces of real–valued functions. We define the operator $A : L_2(0,T;W^1_{2,\gamma}(Q)) \to L_2(0,T;W_2^{-1}(Q))$ by the formula

$$\langle Aw, v\rangle = \sum_{1\le i,j\le n} \int\limits_{\Omega_T} A_{ij}(t,x)\partial_j w(t,x)\,\partial_i v(t,x)\,dx\,dt \qquad \forall v \in L_2(0,T;\mathring{W}_2^1(Q)).$$

We introduce the unbounded operator $\partial_t : L_2(\Omega_T) \supset \mathscr{D}(\partial_t) \to L_2(\Omega_T)$ with the domain

$$\mathscr{D}(\partial_t) := \{w \in L_2(0,T;W_2^1(Q)) : \partial_t w \in L_2(\Omega_T),\, w(0,x) = w(T,x)\}. \tag{6}$$

Definition 1. The function $w \in W_\gamma$ is called the generalized solution of problem (1), (2), (4) if it satisfies the operator equation

$$\partial_t w + Aw = f, \qquad w \in W_\gamma, \tag{7}$$

where $W_\gamma := \mathscr{D}(\partial_t) \cap L_2(0,T;W^1_{2,\gamma}(Q))$.

Note that nonlocal conditions bind the values of the unknown function on some parts of boundary with its values on shifts of these parts into domain Ω_T. The above shifts are generated by a certain difference operator. Properties of such difference operators in the spaces $L_2(Q)$ and $W_2^1(Q)$ were studied earlier, see [3, 4]. In this paper we use the above results to formulate the properties of difference operators acting in different function spaces $R_Q : L_2(\Omega_T) \to L_2(\Omega_T)$ and $R_Q : L_2(0,T;\mathring{W}_2^1(Q)) \to L_2(0,T;W^1_{2,\gamma}(Q))$.

We consider the difference operator

$$Ru(t,x) = u(t,x) + a_1 u(t, x_1+1, x_2) + a_{-1} u(t, x_1-1, x_2).$$

This operator corresponds to boundary conditions (2). We define the operator R_Q given by $R_Q = P_Q R I_Q : L_p(\Omega_T) \to L_2(\Omega_T)$. Here $I_Q : L_2(\Omega_T) \to L_2((0,T)\times\mathbb{R}^n)$ is the operator of extension of functions from $L_2(\Omega_T)$ by zero in $(0,T)\times(\mathbb{R}^n\setminus Q)$, the operator $P_Q : L_2((0,T)\times\mathbb{R}^n) \to L_2(\Omega_T)$ is the operator of restriction of functions from $L_2((0,T)\times\mathbb{R}^n)$ to Ω_T. Then for any $u \in L_2(0,T;\mathring{W}_2^1(Q))$ and $w = R_Q u$ we have

$$w|_{x_1=0} = R_Q u|_{x_1=0} = a_1 u|_{x_1=1}, \qquad w|_{x_1=2} = R_Q u|_{x_1=2} = a_{-1} u|_{x_1=1},$$
$$w|_{x_1=1} = R_Q u|_{x_1=1} = u|_{x_1=1}.$$

Thus, if $a_1 = \gamma_1$, $a_{-1} = \gamma_2$, and $u \in L_2(0,T;\mathring{W}_2^1(Q))$, then the function $w = R_Q u \in L_2(0,T;W_2^1(Q))$ satisfies the nonlocal boundary conditions (2), i.e. $R_Q(L_2(0,T;\mathring{W}_2^1(Q))) \subset L_2(0,T;W_{2,\gamma}^1(Q))$, where $\gamma = \{\gamma_1,\gamma_2\}$. Conversely, if $\gamma_1\gamma_2 \neq 1$, it can be proved that $L_2(0,T;W_{2,\gamma}^1(Q)) \subset R_Q(L_2(0,T;\mathring{W}_2^1(Q)))$, see [3, 4]. Therefore, R_Q maps continuously and bijectively $L_2(0,T;\mathring{W}_2^1(Q))$ onto $L_2(0,T;W_{2,\gamma}^1(Q))$. Hence there exists a bounded inverse operator $R_Q^{-1} : L_2(0,T;W_{2,\gamma}^1(Q)) \to L_2(0,T;\mathring{W}_2^1(Q))$. Consequently, instead of equation (7), we can consider the equation

$$\partial_t R_Q u + A R_Q u = f, \qquad u \in W, \tag{8}$$

where $W := \mathscr{D}(\partial_t) \cap L_2(0,T;\mathring{W}_2^1(Q))$.

Definition 2. A linear operator $\Lambda : L_2(\Omega_T) \supset \mathscr{D}(\Lambda) \to L_2(\Omega_T)$ is **monotone** if

$$\langle \Lambda u, u\rangle := \int\limits_{\Omega_T} \Lambda u \cdot u \, dx\, dt \geq 0 \qquad \forall u \in \mathscr{D}(\Lambda).$$

A linear densely defined monotone operator Λ is **maximally monotone** if there is no linear monotone operator that is a strict extension of Λ.

As is known, in reflexive strictly convex with its conjugate spaces, the maximum monotonicity of the operator is equivalent to the condition:

$$\langle \Lambda u, u\rangle \geq 0 \quad \forall u \in \mathscr{D}(\Lambda), \qquad \langle \Lambda^* u, u\rangle \geq 0 \quad \forall u \in \mathscr{D}(\Lambda^*), \tag{9}$$

see Lemme 1.1 [5, Chapter 3]. It is also known that ∂_t with domain (6) is maximally monotone and $\partial_t^* = -\partial_t$, see [5, Chapter 3, sec.2.2].

Lemma 1. *Let $\gamma_1\gamma_2 \neq 1$. Then $\partial_t R_Q : L_2(\Omega_T) \supset W \to L_2(\Omega_T)$ is maximally monotone.*

Proof. Let $R_Q^c = \dfrac{1}{2}(R_Q + R_Q^*)$ and $R_Q^{cs} = \dfrac{1}{2}(R_Q - R_Q^*)$. Note that

$$(R_Q u(t), u(t))_{L_2(Q)} = \big(u(t), R_Q^* u(t)\big)_{L_2(Q)} = \big(R_Q^* u(t), u(t)\big)_{L_2(Q)} = \big(R_Q^c u(t), u(t)\big)_{L_2(Q)}.$$

Moreover,

$$\begin{aligned}\partial_t \left(R_Q u(t), u(t)\right)_{L_2(Q)} &= \left(\partial_t R_Q u(t), u(t)\right)_{L_2(Q)} + \left(R_Q u(t), \partial_t u(t)\right)_{L_2(Q)} \\ &= \left(R_Q \partial_t u(t), u(t)\right)_{L_2(Q)} + \big(u(t), R_Q^* \partial_t u(t)\big)_{L_2(Q)} = 2\left(\partial_t R_Q^c u(t), u(t)\right)_{L_2(Q)}.\end{aligned}$$

Since $\langle \partial_t R_Q^{cs} u, u\rangle = 0$, we obtain

$$\langle \partial_t R_Q u, u\rangle = \langle \partial_t R_Q^c u, u\rangle = \int_0^T \big(\partial_t R_Q^c u(\tau), u(\tau)\big)_{L_2(Q)} \, d\tau = \frac{1}{2}\big(R_Q^c u(t), u(t)\big)_{L_2(Q)}\Big|_{t=0}^T = 0$$

due to $u(T,x) = u(0,x)$. On the other hand,

$$\langle (\partial_t R_Q)^* u, u\rangle = \langle R_Q^* \partial_t^* u, u\rangle = \langle \partial_t^* R_Q^* u, u\rangle = -\langle \partial_t R_Q^* u, u\rangle = -\langle \partial_t R_Q^c u, u\rangle = 0.$$

The condition (9) is fulfilled. □

Lemma 2. *Let $R_Q^c > 0$, functions $A_{ij} \in C^\infty(\mathbb{R}^3)$, $A_{ij}(t,x) = A_{ji}(t,x)\,(i,j = 1,2)$ are 1–periodic in x_1 and T–periodic in t, and inequality* (3) *holds. Then*

$$\langle A R_Q u, u\rangle \geq c_2 \|u\|^2_{L_2(0,T;\mathring{W}_2^1(Q))}, \tag{10}$$

where $c_2 > 0$ does not depend on u.

For the operator $(AR_Q)(t,\cdot)$, similar estimate is proved in [3, 4].

Note that operator $AR_Q : L_2(0,T;\mathring{W}_2^1(Q)) \to L_2((0,T;W_2^{-1}(Q))$ satisfying (10) is **monotone** and **coercive** in terms of [5].

Theorem 1. *Let $|\gamma_1 + \gamma_2| < 2$, functions $A_{ij} \in C^\infty(\mathbb{R}^3)$, $A_{ij}(t,x) = A_{ji}(t,x)\,(i,j = 1,2)$ are 1–periodic in x_1 and T–periodic in t, and inequality* (3) *holds. Then for any $f \in L_2(\Omega_T)$ there exists a unique generalized solution of problem* (1), (2), (4).

Proof. As stated above, the generalized solution of (1), (2), (4) is the function $w = R_Q u$, where u is the solution of equation (8). Note that if $|\gamma_1 + \gamma_2| < 2$, then $\gamma_1\gamma_2 \neq 1$ and $R_Q^c > 0$, see Examples in [3, 4]. According to Lemma 1, the operator $\partial_t R_Q$ is maximally monotone, and according to Lemma 2, the operator AR_Q is monotone and coercive. The conditions of Theorem 1.1 [5, Chapter III, §1] hold. Thus, the solution of equation (8) exists. The uniqueness of this solution follows from (10). □

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